Chapter-15 30 March, 2020 ①

Exercise

Ques. 3. a) $2\mathbb{Z} = \{0, \pm 2, \pm 4, \dots\}$

$3\mathbb{Z} = \{0, \pm 3, \pm 6, \dots\}$

let if possible $\exists$ a ring isomorphism $\phi : 2\mathbb{Z} \to 3\mathbb{Z}$

Clearly ϕ is one-one and onto

$\phi(4) = \phi(2+2) = \phi(2) + \phi(2) = 2\phi(2)$

Also $\phi(4) = \phi(2.2) = \phi(2)\phi(2) = (\phi(2))^2$

So $2\phi(2) = (\phi(2))^2$

put $x = \phi(2)$

$\Rightarrow$ the equation $x^2 - 2x = 0$ has a solution in $3\mathbb{Z}$

Now $x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0$

$\Rightarrow x = 0$ or 2

$\Rightarrow \phi(2) = 0$ or $\phi(2) = 2$ but $2 \notin 3\mathbb{Z}$

So $\phi(2) = 0 \Rightarrow \phi(2x) = 2\phi(x) = 0 \quad \forall x \in \mathbb{Z}$

$\Rightarrow \phi(2x) = 0 \quad \forall x \in \mathbb{Z}$ but ϕ is not one-to-one

Hence, $x^2 - 2x = 0$ has no solution in $3\mathbb{Z}$

So $\nexists$ any isomorphism from $2\mathbb{Z} \to 3\mathbb{Z}$

So $2\mathbb{Z} \ncong 3\mathbb{Z}$ (as rings)

b) let if possible $2\mathbb{Z} \cong 4\mathbb{Z}$ as rings and let

$\phi : 2\mathbb{Z} \to 4\mathbb{Z}$ is an isomorphism

So $\phi(4) = \phi(2+2) = 2\phi(2)$

$\phi(4) = \phi(2.2) = (\phi(2))^2$

$\Rightarrow 2\phi(2) = (\phi(2))^2$ put $\phi(2) = x$

$\Rightarrow x^2 - 2x = 0$ has solution in $4\mathbb{Z}$

but $x = 0$ or $x = 2$ and $2 \notin 4\mathbb{Z}$

and $4x = 0$ then $\phi(2) = 0$

$\Rightarrow \phi(x) = 0 \quad \forall x$

then ϕ is not one-one

So $2\mathbb{Z} \not\cong 4\mathbb{Z}$ (as rings)

Note: $n\mathbb{Z} \cong m\mathbb{Z}$ as groups

but $n\mathbb{Z} \not\cong m\mathbb{Z}$ ($n, m \rightarrow$ distinct positive integer) as rings

Ques 12 Let $\phi: \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ be a ring homomorphism

To Prove: $\phi(1)$ is an idempotent

Now $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1) = (\phi(1))^2$

So $\phi(1)$ is idempotent.

Note: Idempotent element of a ring: Let $(R, +, \cdot)$ be a ring. An element $a \in R$ is s.t.b idempotent if $\boxed{a^2 = a}$.

Ques 14 Ring homomorphisms from $\mathbb{Z}_6 \rightarrow \mathbb{Z}_6$

Let $\phi: \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$ be a ring homomorphism

(i) ϕ is defined once $\phi(1)$ is defined and

$\phi(m) = m\,\phi(1)$

(ii) $|\phi(1)| \mid |1| \Rightarrow |\phi(1)| \mid 6$

So $|\phi(1)| = 1, 2, 3$ or 6

$\Rightarrow \phi(1) = 0, 3, 2, 4, 1$ or 5

Also $\phi(1) = \phi(1)\phi(1) \Rightarrow \phi(1)$ is idempotent.

So $0^2 = 0$, $1^2 = 1$, $2^2 = 4 \neq 2$, $3^2 = 3$, $4^2 = 4$,

$5^2 = 1 \neq 5$

So $\phi(1) = 0, 1, 3$ or 4

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So $\phi_1(1) = 0$

ring homomorphisms

$\phi_1(x) = x\,\phi_1(1) = 0 \;\forall x$

$\phi_2(1) = 1$	$\phi_3(1) = 3$	$\phi_4(1) = 4$
$\phi_2(x) = x \cdot 1$	$\phi_3(x) = 3 \cdot x = 3x$	$\phi_4(x) = 4x$
$\phi_2 = I$		

b) let $\phi: \mathbb{Z}_{20} \to \mathbb{Z}_{30}$ be ring homomorphism

(i) $\phi(x) = x\,\phi(1)$

(ii) $|\phi(1)| \mid |1|$ and $|\phi(1)| \mid |\mathbb{Z}_{30}|$

$\Rightarrow |\phi(1)| \mid 20$ and $|\phi(1)| \mid 30$

$\Rightarrow |\phi(1)| \mid 10 = \gcd(20, 30)$

So $|\phi(1)| = 1, 2, 5$ or 10

$\Rightarrow \phi(1) = 0, 15, 6, 24, 12, 18, 3, 27$

~~So or 10~~ $(\phi(1))^2 = \phi(1)\,\phi(1) = \phi(1)$

So $0^2 = 0 \quad 15^2 = 15 \quad 6^2 = 6 \quad 24^2 = 6 \neq 24$

$3^2 = 9 \neq 3, \; 27^2 = 9 \neq 27, \; 21^2 = 21$

So $\phi(1) = 0$	$\phi_2(1) = 15$	$\phi_3(1) = 6$	$\phi_4(1) = 21$
$\phi_1(x) = 0$	$\phi_2(x) = 15x$	$\phi_3(x) = 6x$	$\phi_4(x) = 21x$

3 ring homomorphisms

(15) let $\phi : \mathbb{Z} \to \mathbb{Z}$ be a ring homomorphisms

So $\phi(m) = m\phi(1)$

ϕ is defined once $\phi(1)$ is defined

$(\phi(1))^2 = \phi(1)$

$\Rightarrow \phi(1)$ is idempotent

$\phi(1) = x$

$\Rightarrow x^2 - x = 0 \qquad x \in \mathbb{Z}$

$\Rightarrow x = 0, 1 \qquad \mathbb{Z}$ is an integral domain

So $\phi(1) = 0$ or $\phi(1) = 1$

$\phi_1(1) = 0$	$\phi_2(1) = 1$
$\phi_1(x) = x\,\phi_1(1) = 0$	$\phi_2(x) = x.1 = x$
So $\phi_1(x) = 0$	$\Rightarrow \phi_2 =$ Identity map

So there are only 2 ring homomorphisms from $\mathbb{Z} \to \mathbb{Z}$ namely

$\phi : \mathbb{Z} \to \mathbb{Z}$, $\phi(x) = 0 \ \forall x$ and $\phi : \mathbb{Z} \to \mathbb{Z}$, $\phi(x) = x$ } identity map

(29) Given: R & S are commutative rings with unity

$\phi : R \to S$ is an onto homomorphism

Char $R \neq 0$ let Char $R = m$

To Prove: Char S divides char R

We shall prove $ms = 0' \ \forall s \in S$

Now $s \in S$ and $\phi : R \to S$ is onto

$\Rightarrow \exists\, r \in R$ s.t $\phi(r) = s$

Since char $R = m$ $\Rightarrow mr = 0$

$\Rightarrow \phi(mr) = \phi(0) = 0'$

$\Rightarrow m\,\phi(r) = 0'$

$\Rightarrow ms = 0'$

$\Rightarrow$ char S is finite

and char S divides char R.

(30) Given: R is a commutative ring and

Char $R = p \rightarrow$ prime mapping

$\phi : R \rightarrow R$ defined by

$\phi(x) = x^p$

To Prove: ϕ is a ring homomorphism.

(i) $\phi(x+y) = \phi(x) + \phi(y) \quad \forall\, x, y \in R$

(ii) $\phi(xy) = \phi(x)\,\phi(y)$

Do.

Imp.

(32) Given: let F is a field and R is a ring and $|R| > 1$

$\phi : f \rightarrow R$ is an onto homomorphism

To Prove: ϕ is an isomorphism

Since ϕ is onto and homomorphism

We need to prove ϕ is one-to-one

Now $\phi : F \rightarrow R$ is a homomorphism

let $K = \text{Ker}\,\phi$ then K is an ideal of F

But F is a field

F has only two ideals $\{0\}$ and F

If $K = F$, ϕ is onto. By Fundamental Theorem

$$\frac{F}{K} \cong \frac{F}{F} \cong R$$

But $\frac{F}{F} \cong \{0\} \Rightarrow R \cong \{0\}$

But $R \not\cong \{0\}$ as $|R| > 1$ (R has atleast two elements)

Hence, $K = \text{Kernel } \phi = \{0\}$

$\Rightarrow \phi$ is one-one

Hence, ϕ is an isomorphism.

Note: Field has only trivial ideals.

Ques. 45 D is an Integral Domain and F is field of quotients of D

$$F = \left\{ [a/b] : \begin{array}{l} a, b \in D, b \neq 0 \\ \text{and } [a/b] = [c/d] \text{ iff } ad = bc \end{array} \right\}$$ is

E is a field containing D

To Prove: E contains a subfield isomorphic to F

Define a map $\phi : F \to E$ by

$$\phi([a/b]) = ab^{-1} \qquad \left(\begin{array}{l} ab^{-1} \in E \\ \because a, b \in D \\ \text{and } D \subseteq E \end{array} \right)$$

Claim: (i) ϕ is well defined & one-one

(ii) ϕ is homomorphism $\phi\left([a/b] + [c/d]\right) = \phi([a/b]) + \phi[c/d]$

$$\phi([a/b][c/d]) = (\phi([a/b]))\, \phi([c/d])$$

So ϕ is one-one homo. So it must be onto upto homomorphic image

i.e $F \cong \phi(F)$ and $\phi(F)$ is a subfield of E.

Dimension of Subspaces

31 March, 2020

Theorem 1.11: Let W be a subspace of a finite-dimensional vector space V. Then W is finite dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

Proof: Let W be a subspace of a finite dimensional vector space V

If $W = \{0\}$, $\dim W = 0 < \dim V$, W is finite dimensional vector space

If $W \neq \{0\}$, $x_1 \in W$

Choose another vector $x_2 \in W$ such that $\{x_1, x_2\}$ is linearly independent.

Continue this process to get a linearly independent subset $\{x_1, x_2, \ldots x_n\}$ of W

Clearly, $\{x_1, x_2, \ldots, x_n\}$ is a subset of linearly independent set of V. If $\{x_1, x_2, \ldots, x_n\}$ span W, $\{x_1, x_2, \ldots x_n\}$ is the basis for W.

$$\dim W = K \leq \dim V$$

If $\{x_1, x_2, \ldots x_K\}$ is not the basis for W, $\exists$ a subset H of the generating G such that $\{x_1, x_2, \ldots x_K\} \cup H$ generates W and hence form the basis for W

Clearly, $\operatorname{span}(\{x_1, x_2, \ldots x_K\} \cup H) \subseteq \operatorname{Span}(\beta)$ (β is basis for V)

$$\Rightarrow \dim W = |L \cup H| \leq |\beta|$$

$$\Rightarrow \dim W \leq \dim V$$

Hence, W is finite dimensional.

Now $\dim W = \dim V$, $\dim W = k$

$\Rightarrow \{x_1, x_2, \dots x_k\}$ is the basis for W

$\Rightarrow \{x_1, x_2, \dots x_k\}$ is linearly independent set of V

$\exists$ a subset H of the generating set G of V containing $k-k = 0$ vectors such that $L \cup H = L$ generates V.

Hence, L is the basis for V

$$\Rightarrow W = \mathrm{Span}(L) = V$$

i.e $\boxed{W = V}$

<u>example</u> 17

let $W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 : a_1 + a_3 + a_5 = 0, a_2 = a_4\}$

Show that W is a $\boxed{\text{subspace}}$ of F^5. Find its basis.

<u>Sol</u>: $(0,0,0,0,0) \in W$ $\quad (\because 0+0+0=0,\ 0=0)$

$\Rightarrow W \neq \phi$ and $W \subseteq F^5$

let $x, y \in W$

$x = (a_1, a_2, a_3, a_4, a_5)$ $\quad a_1 + a_3 + a_5 = 0$, $a_2 = a_4$

$y = (b_1, b_2, b_3, b_4, b_5)$ $\quad b_1 + b_3 + b_5 = 0$, $b_2 = b_4$

<u>Now</u> $x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$

$$(a_1 + b_1) + (a_3 + b_3) + (a_5 + b_5)$$
$$= (a_1 + a_3 + a_5) + (b_1 + b_3 + b_5)$$
$$= 0 + 0$$
$$= 0$$

and $a_2 - b_2 = a_4 - b_4$ $\quad (\because a_2 = a_4$ and $b_2 = b_4)$

$\Rightarrow x+y \in W$

Now $\alpha x = \alpha(a_1, a_2, a_3, a_4, a_5)$

$= (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4, \alpha a_5)$

$\alpha a_1 + \alpha a_3 + \alpha a_5 = \alpha(a_1 + a_3 + a_5)$

$= \alpha \cdot 0$

$= 0$

and $\alpha a_2 = \alpha a_4 \quad (\because a_2 = a_4)$

$\Rightarrow \alpha x \in W$

Since, $x+y \in W$ and $\alpha x \in W$

$\therefore W$ is a $\boxed{\text{subspace of } F^5}$

Let $x \in W$

i.e $x = (a_1, a_2, a_3, a_4, a_5) \in F^5$

where $a_1 + a_3 + a_5 = 0 \Rightarrow a_1 = -a_3 - a_5$

$a_2 = a_4$

i.e $x = (-a_3 - a_5, a_2, a_3, a_2, a_5)$

$= (0,1,0,1,0)a_2 + (-1,0,1,0,0)a_3 + (-1,0,0,0,1)a_5$

$\Rightarrow W = \text{Span}\{(0,1,0,1,0), (-1,0,1,0,0), (-1,0,0,0,1)\}$

Clearly, $\{(0,1,0,1,0), (-1,0,1,0,0), (-1,0,0,0,1)\}$ is linearly independent set.

Hence, $\{(0,1,0,1,0), (-1,0,1,0,0), (-1,0,0,0,1)\}$ forms the basis for W.

and hence $\boxed{\dim(W) = 3}$

<u>example 18</u>: Show that the set of diagonal $n \times n$ matrices is a subspace W of $M_{n\times n}(F)$. Find its basis

<u>Sol</u>:

$$W = \left\{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_n \end{pmatrix} \middle| \; a_i \in F, \; 1 \le i \le n \right\}$$

$$\because \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{n\times n} \in W$$

$\Rightarrow W \neq \phi$ and $W \subseteq M_{n\times n}(F)$

let $x, y \in W$

$$x = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & a_2 & & \vdots \\ 0 & \cdots & \ddots & a_n \end{pmatrix}_{n\times n}, \quad y = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & b_2 & & \vdots \\ 0 & \cdots & \ddots & b_n \end{pmatrix}_{n\times n}$$

Now,

$$x + y = \begin{pmatrix} a_1 + b_1 & \cdots & & 0 \\ \vdots & a_2 + b_2 & & \vdots \\ & & \ddots & \\ 0 & \cdots & & a_n + b_n \end{pmatrix}_{n\times n} \in W$$

$\Rightarrow x + y \in W$

let $\alpha \in F$, and $x \in W$

$$\alpha x = \alpha \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & a_2 & \vdots \\ 0 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \cdots & 0 \\ \vdots & \alpha a_2 & \vdots \\ 0 & \cdots & \alpha a_n \end{pmatrix}_{n\times n} \in W$$

$\Rightarrow \alpha x \in W$

Since $x + y \in W$ and $\alpha x \in W$

$\therefore W$ is a $\boxed{\text{subspace}}$ of $M_{n\times n}(F)$

let $A \in W$

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & a_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n \end{pmatrix} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & a_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + \cdots\cdots + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_n \end{pmatrix}$$

$$= a_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

$\Rightarrow A = a_1 E^{11} + a_2 E^{22} + \cdots + a_n E^{nn}$

where $E^{ij} = \begin{cases} 1 & \text{, at } i^{th} \text{ row and } j^{th} \text{ column} \\ 0 & \text{, at } i^{th} \text{ row and } j^{th} \text{ column} \end{cases}$

$$\Rightarrow \boxed{A = \text{Span}\left\{ E^{ij} \mid 1 \le i, j \le n \;,\; E^{ij} = \begin{cases} 0 & \text{, when } i \ne j \\ 1 & \text{, when } i = j \end{cases} \right\}}$$

<u>Claim:</u> $\left\{ E^{ij} \mid E \in M_{n\times n}(F) \;,\; E^{ij} = \begin{cases} 0 & \text{, at } i^{th} \text{ row \& } j^{th} \text{ column} \\ 1 & \text{, at } i^{th} \text{ row \& } j^{th} \text{ column} \end{cases} \right\}$ $1 \le i \lneq j \le n$

is linearly independent.

Let $\alpha_1, \alpha_2, \ldots \alpha_n$ be the scalars such that

$\alpha_1 E^{11} + \alpha_2 E^{22} + \cdots + \alpha_n E^{nn} = 0$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \cdots & \cdots & 0 \\ \vdots & \alpha_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \alpha_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

$\Rightarrow \alpha_i = 0 \quad \forall i \quad 1 \le i \le n$

$\Rightarrow E^{ii} \mid E \in M_{n\times n}(F) \;,\; E^{ij} = \begin{cases} 0 & \text{when } i \ne j \\ 1 & \text{when } i = j \end{cases}$

is <u>linearly independent</u>

Hence, the set is $\boxed{\text{basis}}$ for W.

01 April, 2020

example 19: Show that the set of symmetric $n \times n$ matrices is a subspace of W of $M_{n\times n}(F)$.

Find the basis for W.

Sol: $W = \{A \in M_{n\times n}(F) \mid A = A^t\}$

Claim: W forms $\boxed{\text{subspace}}$ of $M_{n\times n}(F)$

i) $A, B \in W$, $A + B \in W$

ii) $A \in W$, $\alpha A \in W$

Do.

Consider a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \underline{b = c}.$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider a 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$\Rightarrow b = d$$
$$c = g$$
$$f = h$$

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$$\Rightarrow \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Claim: $\cancel{B = \{ A^{ij}}$

$B = \{ A^{ij} \in M_{n\times n}(F) \mid a_{ij} = 1$, at i^{th} row & j^{th} column
$= 1$ j^{th} row & i^{th} column,
$1 \le i,j \le n \}$

is the $\boxed{\text{basis}}$ for W.

let $A \in W$

$\Rightarrow A = A^t$

$\Rightarrow [a_{ij}]_{n\times n} = [a_{ji}]_{n\times n} \quad \forall\, a_{ij} \in A$

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & b_1 & b_2 & \cdots & b_{n-1} \\ a_3 & & & & \\ \vdots & & & & \\ a_n & b_{n-1} & \cdots & & z_n \end{pmatrix} = a_1\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} +$$

$$a_2\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} + \cdots + z_n\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

$$= a_1 A^{11} + a_2 A^{12} + \cdots + z_n A^{nn}$$

$\Rightarrow A \in \text{Span}(B)$

$\Rightarrow \boxed{W = \text{Span}(B)}$

Claim: β is linearly independent.

Let $\alpha_{11}, \alpha_{12}, \ldots \alpha_{nn}$ be scalars s.t

$$\alpha_{11} A^{11} + \alpha_{12} A^{12} + \ldots + \alpha_{nn} A^{nn} = 0$$

$$\Rightarrow \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$\Rightarrow \alpha_{ij} = 0 \quad \forall\, i, j \quad 1 \leq i, j \leq n$$

$\Rightarrow \beta$ is linearly independent

$\therefore \beta$ is the $\boxed{\text{basis}}$ for W.

$$\therefore \dim W = n + (n-1) + \ldots + 1$$

$$= \frac{n(n+1)}{2}$$

Corollary: If W is a subspace of finite dimensional vector space V, then any basis for W can be extended to a basis for V.

Proof: Let S be a basis for W

Since S is a linearly independent subset of V,

hence, by the result that every linearly independent subset of V can be extended to a basis for V.

Hence Proved.

example 20 : Show that the set of all polynomials of the form $a_{18}x^{18} + a_{16}x^{16} + \dots + a_2x^2 + a_0$ where $a_{18}, a_{16}, \dots a_2, a_0 \in F$ is a subspace W of $P_{18}(F)$. Show that the basis for W is $\{1, x^2, x^4, \dots x^{16}, x^{18}\}$

Sol^n: $W = \{a_{18}x^{18} + \dots a_2x^2 + a_0 \mid a_i \in F, 0 \le i \le 18\}$

Claim: W is a $\boxed{\text{subspace}}$ of $P_{18}(F)$

i) $W \ne \phi$

ii) $f(x), g(x) \in W$, then $f(x) + g(x) \in W$

iii) $f(x) \in W$ and $\alpha \in F$, $\alpha f(x) \in W$.

Do

Let $f(x) = a_{18}x^{18} + \dots + a_2x^2 + a_0$

$f(x) = \alpha_0 x^{18} + \alpha_1 x^{16} + \dots + \alpha_8 x^2 + \alpha_9$

$\alpha_0 = a_{18}$, $\alpha_1 = a_{16}$, $\dots$ $\alpha_8 = a_2$, $\alpha_9 = a_0$

$\Rightarrow f(x) \in \text{Span}\{x^{18}, x^{16}, \dots x^2, 1\}$

Claim: $\{1, x^2, \dots x^{16}, x^{18}\}$ is linearly independent.

Let $\alpha_0, \alpha_1, \dots \alpha_9$ be the scalars s.t

$$\alpha_0 + \alpha_1 x^2 + \dots + \alpha_8 x^{16} + \alpha_9 x^{18} = 0$$

equating constants & coefficients

$$\alpha_i = 0 \quad \forall i, \ 0 \le i \le 9$$

$\Rightarrow \{1, x^2, x^4, \dots x^{18}\}$ is $\boxed{\text{linearly independent}}$

$\therefore f(x) \in \text{Span}\{1, x^2, x^4, \dots x^{16}, x^{18}\}$

hence, basis for $W = \{1, x^2, x^4, \dots x^{16}, x^{18}\}$

Chapter-2 ~~31 March, 2020~~ ①

Linear Transformations and Matrices

03 April, 2020

Def^n: let V and W be vectorspaces over field F. A function $T: V \to W$ is a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

a) $T(x+y) = T(x) + T(y)$

b) $T(cx) = c\,T(x)$

Note: i) T is $\boxed{\text{linear}}$ if and only if $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$

ii) if T is $\boxed{\text{linear}}$, then $T(0) = 0$

$$\begin{aligned} T(0) &= T(x-x) \\ &= T(x) + T(-x) \qquad (\because T \text{ is linear}) \\ &= T(x) - T(x) \\ &= 0 \end{aligned}$$

$\Rightarrow T(0) = 0$

iii) If T is $\boxed{\text{linear}}$, then $T(x-y) = T(x) - T(y)$ for all $x, y \in V$

iv T is $\boxed{\text{linear}}$ if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

$$\begin{aligned} T\left(\sum_{i=1}^{n} a_i x_i\right) &= T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= T(a_1x_1) + T(a_2x_2) + \cdots + T(a_nx_n) \qquad (\because T \text{ is linear by (i)}) \\ &= a_1T(x_1) + a_2T(x_2) + \cdots + a_nT(x_n) \qquad (\text{by } i) \\ &= \sum_{i=1}^{n} a_i T(x_i) \end{aligned}$$

example 1: Define $T : R^2 \to R^2$ by

$$T(a_1, a_2) = (2a_1 + a_2, a_1)$$

Show that T is linear transformation.

Sol^n: let $x, y \in R^2$

$$x = (a_1, a_2), \quad y = (b_1, b_2)$$

$$x = y$$

$$\Rightarrow (a_1, a_2) = (b_1, b_2)$$

$$\Rightarrow a_1 = b_1 \text{ and } a_2 = b_2$$

$$\Rightarrow 2a_1 = 2b_1 \text{ and } a_2 = b_2$$

$$\Rightarrow 2a_1 + a_2 = 2b_1 + b_2$$

$$\Rightarrow (2a_1 + a_2, a_1) = (2b_1 + b_2, b_1)$$

$$\Rightarrow T(a_1, a_2) = T(b_1, b_2)$$

$$\Rightarrow T(x) = T(y)$$

$\Rightarrow$ T is well defined.

Consider, $T(\alpha x + y) = T(\alpha(a_1, a_2) + (b_1, b_2))$

$$= T((\alpha a_1, \alpha a_2) + (b_1, b_2))$$

$$= T(\alpha a_1 + b_1, \alpha a_2 + b_2)$$

$$= (2(\alpha a_1 + b_1) + \alpha a_2 + b_2, \alpha a_1 + b_1)$$

$$= (2\alpha a_1 + 2b_1 + \alpha a_2 + b_2, \alpha a_1 + b_1)$$

Also, $\alpha T(x) + T(y) = \alpha T(a_1, a_2) + T(b_1, b_2)$

$$= \alpha(2a_1 + a_2, a_1) + (2b_1 + b_2, b_1)$$

$$= (2\alpha a_1 + \alpha a_2, \alpha a_1) + (2b_1 + b_2, b_1)$$

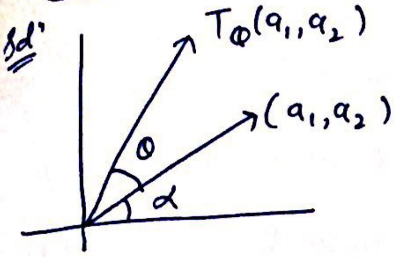
$$= (2\alpha a_1 + \alpha a_2 + 2b_1 + b_2, \alpha a_1 + b_1)$$

$$\therefore T(\alpha x + y) = \alpha T(x) + T(y)$$

$\therefore$ T is $\boxed{\text{linear transformation}}$.

example 2: For any angle θ, define $T_\theta : R^2 \to R^2$ by the rule $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise. Prove that T_θ is linear transformation.

sol:



$T_\theta : R^2 \to R^2$ such that

$T_\theta(a_1, a_2) = T_\theta(r\cos\alpha, r\sin\alpha)$

$T_\theta(a_1, a_2) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$

$$T_\theta(a_1,a_2) = (r(\cos\theta\cos\alpha - \sin\theta\sin\alpha), r(\sin\theta\cos\alpha + \cos\theta\sin\alpha))$$

$$\boxed{T_\theta(a_1,a_2) = (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)}$$

where $a_1 = r\cos\alpha$ and $a_2 = r\sin\alpha$ and $r = \sqrt{a_1^2 + a_2^2}$

Now let $x, y \in R^2$ such that

$$x = y$$

$$\Rightarrow (a_1, a_2) = (b_1, b_2)$$

$$\Rightarrow a_1 = b_1,\ a_2 = b_2$$

$$\Rightarrow a_1\cos\theta = b_1\cos\theta,\quad a_2\cos\theta = b_2\cos\theta$$

$$a_1\sin\theta = b_1\sin\theta,\quad a_2\sin\theta = b_2\sin\theta$$

$$\Rightarrow a_1\cos\theta - a_2\sin\theta = b_1\cos\theta - b_2\sin\theta$$

$$\Rightarrow T_\theta(a_1,a_2) = T_\theta(b_1,b_2)$$

$\Rightarrow T_\theta$ is well defined.

Consider $T_\theta(\alpha x + y) = T_\theta(\alpha(a_1,a_2) + (b_1,b_2))$

$= T_\theta((\alpha a_1, \alpha a_2) + (b_1, b_2))$

$= T_\theta(\alpha a_1 + b_1, \alpha a_2 + b_2)$

$= \big((\alpha a_1 + b_1)\cos\theta - (\alpha a_2 + b_2)\sin\theta,\ (\alpha a_1 + b_1)\sin\theta + (\alpha a_2 + b_2)\cos\theta\big)$

$$\Rightarrow T_\theta(\alpha x+y) = (\alpha(a_1\cos\theta - a_2\sin\theta) + b_1\cos\theta - b_2\sin\theta,$$
$$\alpha(a_1\sin\theta + a_2\cos\theta) + b_1\sin\theta + b_2\cos\theta)$$

Now,

$$\alpha T_\theta(x) + T_\theta(y) = \alpha T_\theta(a_1,a_2) + T_\theta(b_1,b_2)$$
$$= (\alpha(a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta) + (b_1\cos\theta - b_2\sin\theta,\ b_1\sin\theta + b_2\cos\theta))$$
$$= (\alpha(a_1\cos\theta - a_2\sin\theta) + b_1\cos\theta - b_2\sin\theta,\ \alpha(a_1\sin\theta + a_2\cos\theta) + b_1\sin\theta + b_2\cos\theta)$$
$$= T_\theta(\alpha x+y)$$

$$\therefore \boxed{T_\theta(\alpha x+y) = \alpha T_\theta(x) + T_\theta(y)}$$

$\Rightarrow T_\theta$ is a linear transformation.

Example 3: Define $T: R^2 \to R^2$ by

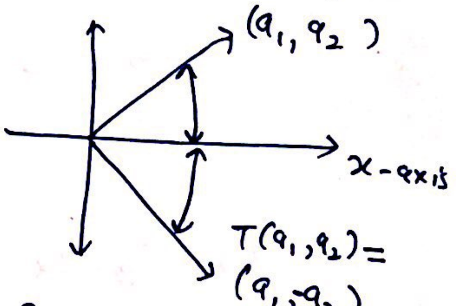
$$T(a_1,a_2) = (a_1,-a_2)$$

T is called the reflection about x-axis.
Show that T is linear transformation.

Sol: Claim: i) T is well defined

ii) $T(\alpha x+y) = \alpha T(x) + T(y)$

Do.



Reflection

Example 4: Define $T: R^2 \to R^2$ by

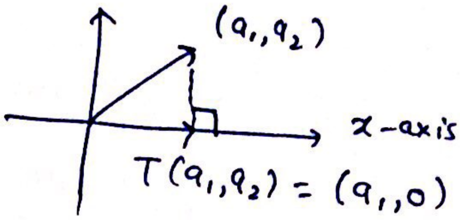
$$T(a_1,a_2) = (a_1,0)$$

T is called the projection on the x-axis.

Sol^n: Claim: i) T is well defined

ii) $T(\alpha x + y) = \alpha T(x) + T(y)$, for $x, y \in R^2$

Prove!



example 5: Define $T : M_{m\times n}(F) \to M_{n\times m}(F)$ by

$$T(A) = A^T , \text{ where } A^T \text{ is the transpose of } A.$$

Show that T is linear transformation.

Sol^n Prove!

example 6: Define $T : P_n(R) \to P_{n-1}(R)$ by

$$T(f(x)) = f'(x)$$

where $f'(x)$ denote the derivative of $f(x)$.

Show that T is linear transformation.

Sol^n. Claim: i) T is well defined

ii) $T(\alpha f(x) + g(x)) = \alpha T(f(x)) + T(g(x))$

Prove!

example 7: let $V = C(R)$ be the vectorspace of continuous real valued functions on R. let $a, b \in R$, $a < b$

Define $T : V \to R$ by

$$T(f) = \int_a^b f(t)\,dt \quad \forall f \in V$$

Show that T is linear transformation.

Sol^n: Prove!!

Defⁿ: Identity Transformation: For a vector space V over F, identity transformation $I_V : V \to V$ is

$$I_V(x) = x \quad \text{for all } x \in V.$$

Defⁿ: Zero Transformation: For a vectorspace V and W over F, Zero transformation $T_0 : V \to W$ is

$$T_0(x) = 0 \quad \text{for all } x \in V.$$

Defⁿ: Null space: let V and W be the two vectorspaces over field F, then the null space (or Kernel) of T is defined as

$$N(T) = \{v \in V \mid T(v) = 0\}$$

i.e It is the set of all vectors v in V such that

$$T(v) = 0$$

Defⁿ: Range of T: Range (or image) $R(T)$ of T is defined to be the subset of W consisting of all images (under T) of vectors in V i.e

$$R(T) = \{T(x) : x \in V\}$$

example 8: Identity Transformation

$$I : V \to V \text{ defined as}$$

$$I(v) = v \quad \forall\, v \in V$$

$$R(T) = \{I(v) : v \in V\} = \{\cancel{x \in}\ \{v : v \in V\} = V$$

$$N(T) = \{v \in V \mid T(v) = 0\} = \{0\}$$

Zero transformation $T_0 : V \to V$ by

$$T_0(v) = 0 \quad \forall\, v \in V$$

$$R(T) = \{T_0(v) : v \in V\} = \{0\}$$

$$N(T) = \{v \in V \mid T_0(v) = 0\} = V$$